Topological invariants of smooth manifolds via classical mechanics and Cauchy - Riemann operator *

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Abstract

We review the construction of two topological invariants of smooth manifolds - Morse and Floer homology and sketch the construction of isomorphism between them, that intertwines with "variation of parameters".

1. Introduction: Topological invariants

Let M be a smooth manifold. An algebraic object $A(M)$ associated to M that does not change under continuous deformations of M is a *topological invariant* of M. This algebraic object $A(M)$ can be, for example, an integer, a group, a ring, a differential graded algebra, etc. Since a coordination is a rule of an assignment a number to some mathematical object, the association $M \mapsto A(M)$ is in fact a "generalized coordination". It translates the problem of distinguishing (classes of homeomorphic) manifolds from topological to algebraic langauge: if $A(M) \neq A(N)$ then $M \neq N$.

There are, in some sense, two ways of constructing topological invariants - direct and indirect way.

If one uses only topology of M to construct topological invariant, then we say that it is obtained in a direct way. This is illustrated in the following classical examples.

Example 1 The simplest example is the number of connected components of M, $\pi_0(M)$ (see Figure 1).

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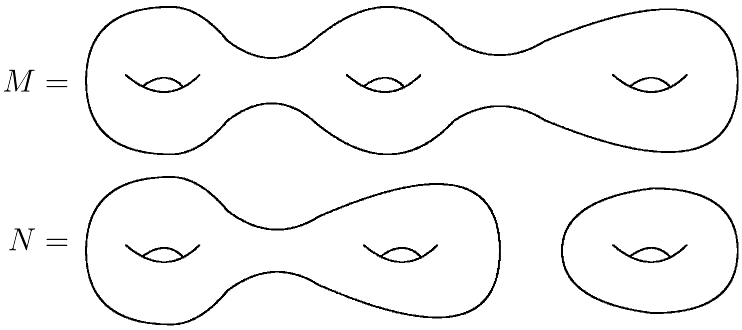

Figure 1: $\pi_0(M) \neq \pi_0(N)$

Example 2 The homotopy classes of based loops in M form a fundamental group $\pi_1(M)$ (see Figure 2).

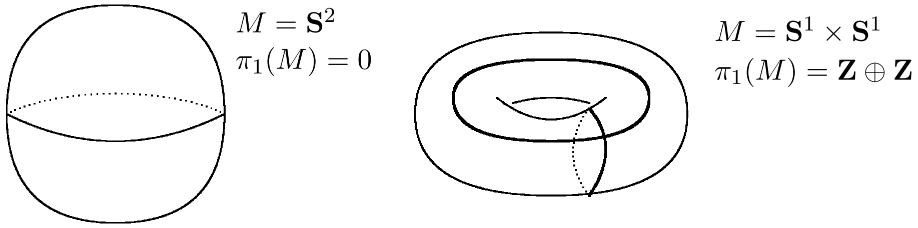

Figure 2: Fundamental group π_1

Example 3 A natural generalization of the Example 1 and Example 2 are higher homotopy groups $\pi_n(M)$, homotopy classes of based $n-$ spheroids (continuous maps $\mathbf{S}^n \mapsto M$).

Example 4 The second natural generalization are homology and cohomology groups, $H_k(M)$ and $H^k(M)$, for $k \geq 0$.

The indirect way to make a topological invariant uses some auxiliary structure (for example, Riemannian metric, symplectic form, smooth function, etc.) to construct an algebraic object. The crucial step of this construction is to prove that the obtained algebraic object is independent of the auxiliary structure, and thus that obtained object is indeed a *topological* invariant.

Example 5 For given smooth manifold M take a continuous function

$$f : M \to \mathbb{R}$$

(as an auxiliary structure). Denote by $N(f)$ the number of its minima (or maxima). If $N(f)$ is strictly greater than zero, for any such function f, then M is compact. The converse is obviously true. Here the auxiliary structure is f, but the claim $N(f) > 0$ is independent on particular choice of f.

The relation between critical points of functions and topology of smooth manifolds (the simplest case is the Example 5) is formalized in *Morse theory.*

2. Morse theory

A smooth function on a smooth manifold, $f : M \to \mathbb{R}$ is called *Morse* if all its critical points are non-degenerate, i.e. if the matrix $d^2 f(p)$ is non - singular for every critical point p. According to classical result called Morse Lemma one can choose coordinates $(x_1, x_2, \ldots, x_n)$ in the neighborhood of p so that

$$f(x_1, x_2, \ldots, x_n) = f(p) - x_1^2 - x_2^2 - \ldots - x_k^2 + x_{k+1}^2 + x_{k+2}^2 + \ldots + x_n^2$$

where k is the number of negative eigenvalues of the matrix $d^2 f(p)$. The number k is independent of the choice of coordinates and it is called Morse index of f at critical point p. An obvious consequence of Morse Lemma is that the critical points of Morse function are isolated.

Suppose that M is compact; in that case the set of critical points of f is finite. Morse chain complex is a $\mathbf{Z}_2$–vector space $CM_*(f)$ generated by the set of these critical points. Here $*$ stands for grading, which is determined by Morse index of f. Fix a Riemannian metric g on M and define the gradient vector field of f, ∇f, to be the g–dual of a differential df, i.e.

$$df(x)(\zeta_x) = g(\zeta_x, \nabla f(x)), \quad x \in M, \ \zeta_x \in T_x M. \tag{1}$$

The number of gradient trajectories that connect two critical points defines the boundary operator on $CM_*(f)$. More precisely, for two critical points p, q of f, let $n(p,q)$ be the number (mod $\mathbf{Z}_2$) of solutions of ordinary differential equation

$$\begin{cases} \dfrac{d\gamma}{dt} + \nabla f(\gamma) = 0 \\ \gamma(-\infty) = p, \ \gamma(+\infty) = q \end{cases} \tag{2}$$

(see Figure 3).

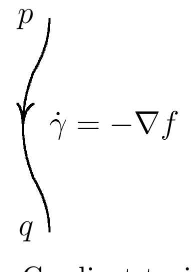

Figure 3: Gradient trajectory

The boundary operator is defined by

$$\partial : CM_*(f) \to CM_{*-1}(f), \qquad \partial(p) := \sum_{q \in \mathrm{Crit}(f)} n(p,q) q.$$

Morse homology groups $HM_*(f)$ are the homology groups of $CM_*(f)$ with respect to ∂, i.e:

$$HM_*(M) := \frac{\operatorname{Ker}\partial}{\operatorname{Im}\partial}$$

The proof of $\partial^2 = 0$ is based on the following cobordism argument: the boundary of one - dimensional manifold of gradient trajectories that connect critical points p and q of Morse indices $*$ and $*-1$ is the union of "broken" trajectories that appear in the definition of ∂.

Morse homology is a topological invariant of M constructed indirectly by means of two auxiliary structures: a Morse function f and a Riemannian metric g. As we said, the crucial step is to prove the independence of an auxiliary structure. For two Morse functions f^α, f^β there exists an isomorphism

$$T^{\alpha\beta} : HM_*(f^\alpha) \to HM_*(f^\beta).$$

It is defined again by counting the numbers of the solutions of an ordinary differential equation. More precisely, for a fixed $R_0 > 0$, let $f_t^{\alpha\beta}$ be a $t-$dependent family of smooth functions such that $f_t^{\alpha\beta} = f_t^\alpha$, for $t \leq -R_0$ and $f_t^{\alpha\beta} = f_t^\beta$, for $t \geq R_0$. Denote by $n^{\alpha\beta}(p_\alpha, p_\beta)$ the number of solution of

$$\begin{cases} \dfrac{d\gamma}{dt} + \nabla f^{\alpha\beta}(\gamma) = 0 \\ \gamma(-\infty) = p^\alpha,\ \gamma(+\infty) = p^\beta. \end{cases} \tag{3}$$

The isomorphism $T^{\alpha\beta}$ is defined on generators by:

$$T^{\alpha\beta}(p_\alpha) := \sum_{p_\beta \in \operatorname{Crit}(f^\beta)} n^{\alpha\beta}(p_\alpha, p_\beta) p^\beta.$$

In a similar way one shows the independence of Riemannian metric. Moreover, the following important theorem holds.

Theorem 1 *Morse homology groups $HM_*(f)$ are isomorphic to singular homology groups $H_*(M; \mathbf{Z}_2)$.*

We refer the reader to [18, 26, 25] for different proofs of Theorem 1. For more details about Morse theory see also [17, 19, 20, 27, 29].

One generalization of Morse theory is *Floer theory*, developed by Andreas Floer in a series of papers [3, 4, 5, 6, 7, 8, 9, 10]. It is based on Witten's work on supersymmetry [29] and Gromov's work on pseudo holomorphic curves [12].

3. Floer thoery

Although at first appears that the natural ambient for Morse theory is Riemannian geometry (as the definition of gradient flow relies upon the presence of Riemannian metric; recall (1)), it turns out that Morse theory is essentially symplectic in its nature. Indeed, recall that for any smooth manifold M its cotangent bundle T^*M carries canonical symplectic structure ω_0 (see [2]). The graph of the differential df of any smooth function $f : M \to \mathbb{R}$

$$\Gamma(df) := \{df(x) \mid x \in M\} \subset T^*M$$

is a Lagrangian submanifold in T^*M, i.e. ω_0 vanishes on its tangent space and its dimension is the half of the dimension of the ambient manifold T^*M. The property of f being Morse can be rephrased as a transversality of $\Gamma(df)$ to the zero–section O_M in T^*M, and the solutions of gradient flow equation (2) are in one-to-one correspondence with the solutions Cauchy–Riemann equation $\bar{\partial}u = 0$ in T^*M with Lagrangian boundary condition (see [6]). This observation generalizes in a following way.

Let P be a symplectic manifold and $L_0, L_1 \subset P$ two Lagrangian submanifolds. Floer chain groups $CF_*(L_0, L_1)$ are $\mathbf{Z}_2$–vector spaces generated by the set $L_0 \cap L_1$. Under certain conditions (see [21, 22] and, for more general results,[11]), Floer homology $HF_*(L_0, L_1)$ for the pair L_0, L_1 is defined as the homology group of $CF_*(L_0, L_1)$. The boundary operator is defined by

$$\delta : CF_*(L_0, L_1) \to CF_{*-1}(L_0, L_1), \quad \delta(x) := \sum_{y \in L_0 \cap L_1} n(x, y)y,$$

where $n(x, y)$ is the number (mod $\mathbf{Z}_2$) of holomorphic discs u that satisfy Cauchy - Riemann equation with Lagrangian boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial s} + J\dfrac{\partial u}{\partial t} = 0 \\ u(s, i) \in L_i, \ i \in \{0, 1\} \\ u(-\infty, t) \equiv x, \ u(+\infty, t) \equiv y, \ x, y \in L_0 \cap L_1 \end{cases} \tag{4}$$

(see Figure 4).

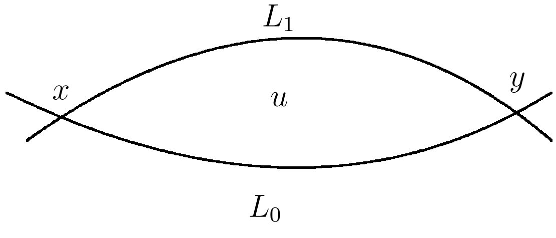

Figure 4: Holomorphic disc u

Although only the almost complex structure J figures in the explicit formula (4) in the definition of δ, the symplectic structure is in fact the most important ingredient in construction of Floer homology.

In particular, let $P = T^*M$ be a cotangent bundle over a compact manifold M, $H : T^*M \to \mathbb{R}$ a smooth Hamiltonian, X_H a corresponding Hamiltonian vector field and ϕ_t^H Hamiltonian flow, i.e. the smooth flow locally given as the solution of

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{5}$$

in canonical coordinates (q, p). If $L_0 = O_M$ is a zero section, and $L_1 = \phi_1^H(L_0)$ a Hamiltonian deformation of L_0, then the system (4) is equivalent to

$$\begin{cases} \dfrac{\partial u}{\partial s} + J\dfrac{\partial u}{\partial t} = JX_H(u) \\ u(s,i) \in L_0 = O_M,\ i \in \{0,1\} \\ u(-\infty,t) = \phi_t^H((\phi_1^H)^{-1})(x), \\ u(+\infty,t) = \phi_t^H((\phi_1^H)^{-1})(y),\ x,y \in O_M \cap \phi_1^H(O_M) \end{cases} \tag{6}$$

so the boundary operator is defined by the number of perturbed holomorphic "tunnels" with boundary on O_M, instead of holomorphic discs (4) (see Figure 5).

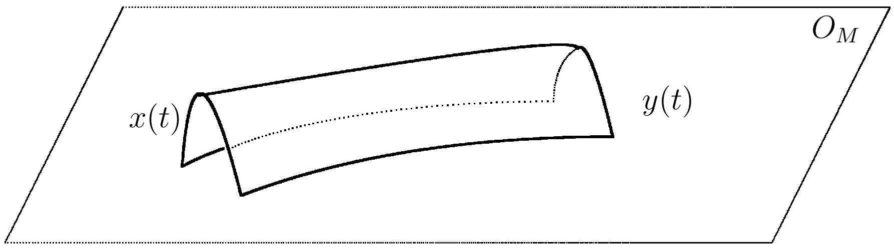

Figure 5: Perturbed holomorphic tunnel u

Note that the equation (6) is inhomogeneous (unlike the homogeneous equation (4)) and that the boundary conditions in (4) and (6) are different. The equivalence of (4) and (6) follows from simple change of variables. It is worthwhile mentioning that the equation (6) can be considered as the gradient flow of a classical Hamiltonian Action Functional

$$\mathcal{A}_H(\gamma) := \int_\gamma p\,dq - H dt$$

(here $p\,dq$ is the Liouville form on T^*M) on the space of paths starting and ending on O_M. Its solutions are two-parameter families of paths and they connect two critical points of $\mathcal{A}_H$ (i.e. two solutions of (5) that start and end on O_M). Hence the Floer theory (in this setting) can be considered as *the infinite dimensional Morse theory of the Action Functional.* It turned to be fruitful to consider the Floer homology groups together with the filtration given by the level sets of $\mathcal{A}_H$. This gives rise to interesting numerical invariants of Lagrangian submanifolds (see [23] for survey and [28], [16] for finite–dimensional analogues based on construction in [15]).

Denote Floer homology groups $HF_*(O_M, \phi_1^H(O_M))$ by $HF_*(H)$. As before, one needs to prove that they are independent on the auxiliary structure, which is in this case a Hamiltonian H. For two Hamiltonians H^α, H^β the corresponding Floer homology groups $HF_*(H^\alpha)$ and $HF_*(H^\beta)$ are isomorphic. The isomorphism is defined by counting the solution of a partial differential equation similar to (6). More precisely, fix $R_0 > 0$. Let $H^{\alpha\beta}(s,t,x)$ be a smooth function such that $H^{\alpha\beta}(s,t,x) = H^\alpha(t,x)$, for $s \leq -R_0$ and $H^{\alpha\beta}(s,t,x) = H^\beta(t,x)$, for $s \geq R_0$. The isomorphism

$$S^{\alpha\beta} : HF_*(H^\alpha) \to HF_*(H^\beta)$$

is defined by

$$S^{\alpha\beta}(x^\alpha) := \sum_{x^\beta} n(x^\alpha, x^\beta) x^\beta$$

where $n(x^\alpha, x^\beta)$ is a number (mod $\mathbf{Z}_2$) of the solutions of the system

$$\begin{cases} \dfrac{\partial u}{\partial s} + J\dfrac{\partial u}{\partial t} = JX_{H^{\alpha\beta}}(u) \\ u(s,i) \in L_0, \ i \in \{0,1\} \\ u(-\infty,t) = \phi_t^{H^\alpha}((\phi_1^{H^\alpha})^{-1})(x^\alpha), \ u(+\infty,t) = \phi_t^{H^\beta}((\phi_1^{H^\beta})^{-1})(x^\beta), \\ x^\alpha \in O_M \cap \phi_1^{H^\alpha}(O_M), \ x^\beta \in O_M \cap \phi_1^{H^\beta}(O_M). \end{cases} \tag{7}$$

Floer [6] proved that Floer homology is isomorphic to the singular homology of M. The isomorphisms between Floer and singular homologies is established though reduction of Floer homology to Morse homology in a following way. Any C^2–small Morse function $f : M \to \mathbb{R}$ can be extended to a Hamiltonian $H_f : T^*M \to \mathbb{R}$ so that the intersection points $O_M \cap \phi_1^{H_f}(O_M)$ are in one-to-one correspondence with critical points of f and the solutions of (3) are in one-to-one correspondence with the solutions of (6) (see [6] for details). Hence we have the isomorphisms

$$H_*(M; \mathbf{Z}_2) \cong HM_*(f) \cong HF_*(H_f).$$

By now we sketched the proofs of independence of parameters and the isomorphism between Morse and Floer homologies. However, we have not proved yet the functoriality of Morse - Floer theory, i.e. the commutativity of the diagram

$$
\begin{array}{ccc}
HF_*(H^\alpha) & \xrightarrow{S^{\alpha\beta}} & HF_*(H^\beta) \\
\uparrow & & \uparrow \\
HM_*(f^\alpha) & \xrightarrow{T^{\alpha\beta}} & HM_*(f^\beta).
\end{array}
\tag{8}
$$

The commutativity of (8) is not obvious because of the fact that the isomorphisms $T^{\alpha\beta}$ and $S^{\alpha\beta}$ are defined by means of two analytically different tasks: ordinary differential equation and elliptic partial differential equation. Motivated by [24], we overcome this difficulty by introducing objects of mixed type that incorporate both gradient flow and Cauchy - Riemman equation. So the isomorphism between Morse and Floer homology is established for any Morse function f on M and any Hamiltonian H on T^*M (not only the special one H_f), by counting the mixed type objects, i.e. pairs (u,γ) of "tunnels" and gradient trajectories

$$
u : (-\infty, 0] \times [0,1] \to T^*M, \quad \gamma : [0,+\infty) \to M
$$

that satisfy the equation

$$
\begin{cases}
\dfrac{d\gamma}{dt} = -\nabla f(\gamma(t)), \\
\dfrac{\partial u}{\partial s} + J\dfrac{\partial u}{\partial t} = X_{\rho_R H}(u) \\
u(\partial([0,+\infty) \times [0,1])) \subset O_M, \\
u(-\infty, t) = x(t), \gamma(+\infty) = p, \\
\gamma(0) = u(0, \frac{1}{2})
\end{cases}
\tag{9}
$$

(see Figure 6). Here $\rho_R : (-\infty, 0] \to \mathbb{R}$ is smooth function such that $\rho_R(t) = 1$, for $t \leq -R - 1$ and $\rho_R(t) = 0$, for $t \geq -R$.

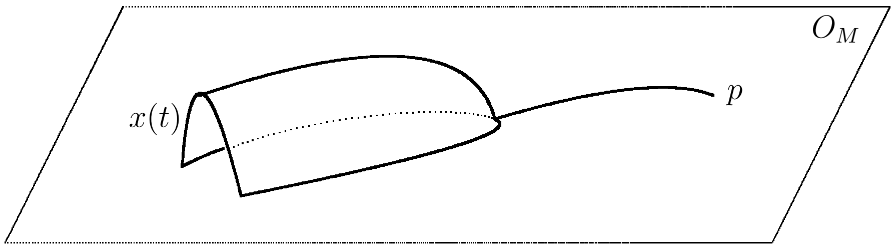

Figure 6: Mixed type object (u,γ)

Using this isomorphism instead of one used in Floer's proof one can prove that the diagram (8) commutes. The proofs of the facts that the homomorphism defined by (9) is well defined, that it is an isomorphism and that the diagram (8) commutes is based on certain cobordism arguments and the fact that the number of points in the boundary of one - dimensional manifold is even. Namely, consider one - dimensional component of a manifold of mixed - type objects as in the Figure 6. One can show that its boundary consists of the two types of "broken" mixed type objects (see Figure 7) and using this fact one can prove the mentioned claims (see [14, 13] for the details).

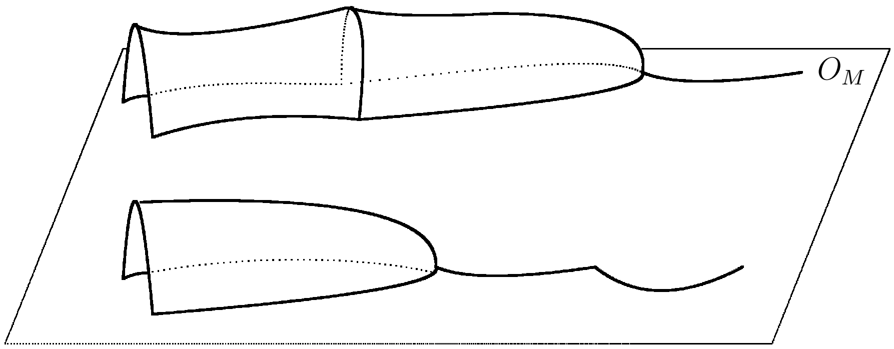

Figure 7: The boundary of one-dimensional manifold of mixed type objects

The generalisation of the construction presented above from cotangent bundles to more general symplectic manifolds does not give isomorphisms, but only homomorphisms (see [1]). In contrast to this, similar question in Floer homology for periodic orbits for more general symplectic manifolds was resolved by Piunikhin, Salamon and Schwarz [24]. Instead of the isomorphisms described above, they constructed the isomorphisms defined by counting the intersection numbers of spaces of perturbed holomorphic cylinders and spaces of gradient trajectories.

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